International Journal of Theoretical Physics, Vol. 39, No. 3, 2000

Absolute Continuity of States on Concrete Logics[†]

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Received December 8, 1999

By concrete logic we mean a quantum logic which is set-representable, and by Vitali–Hahn–Saks logic (VHS-logic) we mean a concrete logic for which the Vitali–Hahn–Saks theorem holds true. In this note we investigate the size of the class of VHS-logics, showing among others that each concrete logic can be concretely enlarged to a VHS-logic as well as to a non-VHS-logic.

1. INTRODUCTION AND BASIC DEFINITIONS

Palko [3] studied VHS-logics, showing that concrete logics may (but need not) be VHS-logics (see also [2]). We strengthen and supplement his results by constructing a proper class of logics which *are* VHS and a proper class of logics which *are not* VHS. Let us review the basic notions we shall use in the sequel.

Definition 1. By *concrete logic* we mean a couple (Ω, Δ), where Ω is a nonempty set and Δ is a set of subsets of Ω satisfying the following properties (the symbols $\cup$ and $\setminus$, respectively, denote the set-theoretic union and complementation):

(i) $\emptyset \in \Delta$.
(ii) $A \in \Delta \Rightarrow \Omega \setminus A \in \Delta$.
(iii) $A_i \in \Delta$ $(i \in \mathbb{N})$, $A_i \cap A_j = \emptyset$ $(i \neq j) \Rightarrow \cup_{i \in \mathbb{N}} A_i \in \Delta$.

(for more details, see [4]).

A concrete logic is therefore a generalization of Boolean σ-algebra of subsets of a set. Obviously, a concrete logic (Ω, Δ) is a Boolean σ-algebra if and only if $A \cap B \in \Delta$ for each $A, B \in \Delta$.

[†]This paper is dedicated to the memory of Prof. Gottfried T. Rüttimann.

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0020-7748/00/0300-0615$18.00/0

Definition 2. By *state* on a concrete logic (Ω, Δ) we mean a function $s: \Delta \to [0, 1]$ satisfying the following properties:

(i) $s(\Omega) = 1$.
(ii) $A_i \in \Delta\ (i \in \mathbb{N}),\ A_i \cap A_j = \emptyset (i \neq j) \Rightarrow s(\cup_{i \in \mathbb{N}} A_i) = \Sigma_{i \in \mathbb{N}}\, s(A_i)$.

In order to introduce the concept of Vitali–Hahn–Saks logic, we need a preliminary definition.

Definition 3. Let s and t be two states on the concrete logic (Ω, Δ). We say that s is *absolutely continuous* with respect to t (in symbols: $s \ll t$) if

$$\forall \varepsilon > 0, \exists \delta > 0: \quad t(A) < \delta \Rightarrow s(A) < \varepsilon$$

Definition 4. A concrete logic (Ω, Δ) is called *Vitali–Hahn–Saks* (VHS) if the Vitali–Hahn–Saks theorem holds true for it. This means: Given $(s_i)_{i \in \mathbb{N}}$ and t states on (Ω, Δ) such that each s_i is absolutely continuous with respect to t and assuming that for each $A \in \Delta$ there exists the limit $\lim_i s_i(A) = s(A)$, then:

(i) s is a state on Δ.
(ii) $s \ll t$.
(iii) The states s_i $(i \in \mathbb{N})$ are *uniformly* continuous with respect to t.

It can be proved that the condition (i) is always satisfied (see [2]). Our consideration therefore reduces to analyze the condition (iii), which is obviously stronger than (ii). We will employ the following definition.

Definition 5. Let (Ω_1, Δ_1) and (Ω_2, Δ_2) be two concrete logics. A mapping $e: \Delta_1 \to \Delta_2$ is called a *concrete enlargement* [and, in this case, (Ω_2, Δ_2) is called a concrete enlargement of (Ω_1, Δ_1)] if the following properties are satisfied:

(i) $e(\emptyset) = \emptyset$.
(ii) $e(\Omega_1 \backslash A_1) = \Omega_2 \backslash e(A_1)$.
(iii) If $A_i \in \Delta_1 (i \in \mathbb{N})$ and $A_i \cap A_j = \emptyset (i \neq j)$, then

$$e(\cup_{i \in \mathbb{N}} A_i) = \cup_{i \in \mathbb{N}} e(A_i)$$

(iv) The mapping e is injective and, moreover, for all $A, B \in \Delta_1$,

$$A \cap B \in \Delta_1 \Leftrightarrow e(A) \cap e(B) \in \Delta_2$$

It should be observed that the intuitive content of Definition 5 is the intention to completely reproduce Δ_1 in Δ_2. For instance, according to Definition 5, it is impossible to enlarge a non-Boolean concrete logic to a Boolean one.

2. RESULTS

Our first result says that there is a proper class of non-VHS-logics. Moreover, their "intrinsic structure" can be in a sense as general as the intrinsic structure of arbitrary logics. This result will be seen in a rather interesting interplay with the next one.

Theorem 1. Each concrete logic (Ω_1, Δ_1) can be concretely enlarged to a non-VHS-logic. Moreover, we can find a concrete non-VHS enlargement (Ω_2, Δ_2) such that card $\Delta_2 = \max(\text{card } \Delta_1, \aleph_0)$.

Proof. Let (Ω_1, Δ_1) be a concrete logic. If card$(\Delta_1) = 2$, then the result is an easy consequence of the existence of non-VHS logics. Such logics exist, as we show in the sequel. Suppose card $(\Delta_1) > 2$. Denote by Ω_2 the Cartesian product, $\Omega_2 = \Pi_{i \in \mathbb{N}} \Omega_1 = \Omega_1 \times \Omega_1 \times \Omega_1 \times \ldots$, of countably many copies of Ω_1, and denote by Δ_2 the subcollection of subsets of Ω_2 ($\Delta_2 \subseteq \exp \Omega_2$) defined in the following way:

$$X \in \Delta_2 \Leftrightarrow X = \Pi_{i \in \mathbb{N}} A_i \text{ with } A_i \in \Delta_1 \text{ for each } i \in \mathbb{N} \text{ and}$$

$$\text{card}(\{i \in \mathbb{N}: A_i \neq \Omega_1\}) \leq 1$$

It is easy to check that (Ω_2, Δ_2) is a concrete logic and that the mapping e: $\Delta_1 \to \Delta_2$ defined by putting

$$e: A \to A \times \Omega_1 \times \Omega_1 \times \cdots$$

is a concrete enlargement. We claim that (Ω_2, Δ_2) is not VHS. To show it, let us first note that for each $X = \Pi_{i \in \mathbb{N}} A_i \in \Delta_2 \backslash \{\emptyset, \Omega_2\}$ there exists a unique $n \in \mathbb{N}$ such that $A_i = \Omega_1$ for each $i \neq n$. In this case we write $X = [A_n, n]$. We will construct a sequence of states on Δ_2, violating the VHS condition. We shall use the technique of [3], Example 2.1: Take a set $C \in \Delta_1 \backslash \{\emptyset, \Omega_1\}$ and pick two points p and q of Ω_1 such that $p \in C$ and $q \notin C$. Define the mappings s and t on Δ_2 in the following way:

$$s(\emptyset) = t(\emptyset) = 0, \qquad s(\Omega_2) = t(\Omega_2) = 1$$

and for $X \in \Delta_2 \backslash \{\emptyset, \Omega_2\}$ with $X = [A_n, n]$, put

$$s([A_n, n]) = \begin{cases} 0 & \text{if} \quad \text{card}(\{p, q\} \cap A_n) = 0 \\ 1/2 & \text{if} \quad \text{card}(\{p, q\} \cap A_n) = 1 \\ 1 & \text{if} \quad \text{card}(\{p, q\} \cap A_n) = 2 \end{cases}$$

$$t([A_n, n]) = \begin{cases} 0 & \text{if} \quad p, q \notin A_n \\ 1/n & \text{if} \quad p \in A_n, \quad q \notin A_n \\ 1 - 1/n & \text{if} \quad p \notin A_n, \quad q \in A_n \\ 1 & \text{if} \quad p, q \in A_n \end{cases}$$

It is easy to verify that s and t are states on Δ_2. Indeed, it is sufficient to observe that for the elements $X = [A_n, n]$, $Y = [A_m, m]$ of Δ_2 we find $X \cap Y = \emptyset$ if and only if $n = m$ and $A_n \cap A_m = \emptyset$.

Finally, let us define a sequence s_i of states on Δ_2 in the following way:

$$s_i(\emptyset) = 0, \qquad s_i(\Omega_2) = 1 \text{ for each } \quad i \in \mathbb{N} \text{ and}$$

$$s_i([A_n, n]) = \begin{cases} s([A_n, n]), & n \leq i \\ t([A_n, n]), & n > i \end{cases}$$

Let us verify that $s_i \ll t$ for each $i \in \mathbb{N}$. Take $i \in \mathbb{N}$ and fix $\varepsilon > 0$. Define $\delta = \min\{\varepsilon, 1/(2i)\}$. If $t(X) < \delta$, we see that $X = [A_n, n]$ with $n > i$. Then $s_i(X) = t(X) < \varepsilon$.

It is easy to check that $\lim_i s_i(X) = s(X)$ for each $X \in \Delta_2$. By the construction, $s \not\ll t$. This completes the proof. ■

Let us now formulate and prove our next result. The result says that the VHS-logics exist in abundance. (Recall that a set S is called non-real measurable if each σ-additive state on the Boolean σ-algebra of all subsets of S lives on a countable set. It is known [5] that there are theories of sets where all sets are non-real measurable. If we assume that all sets are non-real measurable, we use the symbol *non-RM*. Recall also [5] that the least uncountable cardinality $\aleph_1$ is non-real measurable even in the standard ZFC theory.)

Theorem 2 (Non-RM). Each concrete logic can be concretely enlarged to a VHS-logic. Moreover, if (Ω_1, Δ_1) is a concrete logic, we can find a concrete VHS enlargement (Ω_2, Δ_2) such that card $\Omega_2 = \max(\text{card } \Omega_1, \aleph_1)$.

Let us first formulate a simple lemma which will be used in the proof.

Lemma 1. Let (Ω, Δ) be a concrete logic and let $C(\Delta)$ be the center of Δ [i.e., $C(\Delta) = \{A \in \Delta: A \cap B \in \Delta \text{ for each } B \in \Delta\}$]. Suppose that each state on (Ω, Δ) has the following property: For each $B \in \Delta$ there is $A \in C(\Delta)$, $A \subseteq B$, such that $s(B) = s(A)$. Then (Ω, Δ) is a VHS-logic.

The proof of the lemma is elementary.

Let us return to the proof of Theorem 2.

Proof. We shall make use of (an improvement of) the technique of [1]. Let (Ω_1, Δ_1) be a concrete logic. Take an infinite set P such that card $P = \aleph_1$, and consider the set $\Omega_2 = P \times \Omega_1$. For a subset A of Ω_2 and an element $t \in P$, write $A_t = \{y \in \Omega_1: (t, y) \in A\}$. Take the collection $\Delta_2 \subset \exp \Omega_2$ defined in the following way: $A \in \Delta_2 \Leftrightarrow$ the set $\{t \in P: A_t \notin \Delta_1\}$ is at most countable. In order to verify that Δ_2 is a concrete logic, only the closedness under the countable disjoint unions has to be shown, the other conditions being clearly satisfied. If $A_i \in \Delta_2$, $i \in \mathbb{N}$, and $A_i \cap A_j = \emptyset$ for $i \neq j$, then, by definition of Δ_2, there exists a sequence $(C_i)_{i\in\mathbb{N}}$ of countable subsets of P such that for each $i \in \mathbb{N}$, $t \in P\backslash C_i$, we have $(A_i)_t \in \Delta_1$. Putting $C = \cup_{i\in\mathbb{N}} C_i$, the relation $(A_i)_t \in \Delta_1$ is true for any $t \in P\backslash C$. Since the sets A_i are pairwise disjoint, the sets $(A_i)_t$ have the same property. Then $\cup_i (A_i)_t = (\cup_i A_i)_t \in \Delta_1$ for each $t \in P\backslash C$. This means $\cup_i A_i \in \Delta_2$. The hypothesis card $P = \aleph_1$ ensures that the mapping e: $A \to P \times A$ is a concrete enlargement of Ω_1 to Ω_2. It remains to show that (Ω_2, Δ_2) is a VHS-logic.

Let us verify that each state on Δ_2 lives on a countable set. Let s be a state on Δ_2. The mapping $\hat{s}$: $A \to s(A \times \Omega_1)$ is a state on the Boolean σ-algebra exp P of all subsets of P. In view of *non-RM*, there exists a countable subset D of P such that $\hat{s}(D) = 1$. Thus, $s(D \times \Omega_1) = 1$. We can assume that $s(\{d\} \times \Omega_1) > 0$ for any $d \in D$. For each $d \in D$, let us consider the mapping s_d: exp $\Omega_1 \to [0, 1]$ defined as follows:

$$s_d\colon \quad A \to \frac{s(\{d\} \times A)}{s(\{d\} \times \Omega_1)}$$

This mapping is a state on exp Ω_1. In view of *non-RM*, there exists a countable set $C_d \subseteq \Omega_1$ such that $s_d(C_d) = 1$. Put $C = \cup_{d\in D} C_d$ and $E = D \times C$. We have

$$\begin{aligned} s(E) &= s(D \times C) = \sum_{d\in D} s(\{d\} \times C) \\ &= \sum_{d\in D} s(\{d\} \times \Omega_1) = s(D \times \Omega_1) = 1 \end{aligned}$$

We have shown that the state s lives on a countable subset E of Ω_1. Since all countable subsets of Ω_2 belong to Δ_2 (in fact, all singletons of Ω_2 are elements of Δ_2), we have $s(A) = s(A \cap E)$ for any set $A \in \Delta_2$. It suffices to observe that each countable set in Ω_2 belongs to the center of Δ_2 and we complete the proof by applying Lemma 1. ■

The previous result allows us to construct many non-Boolean concrete VHS-logics. We conclude by observing that the property of being VHS is stable, in the class of concrete logics, under the formation of countable products and epimorphic images. This shows, together with the results of [3], that the class of VHS-logics is considerably large.

ACKNOWLEDGMENTS

This research was carried out while the author was on a one-year study leave in Prague. The author would like to gratefully acknowledge the financial support of CNR, Italy, which made the stay possible. She also gratefully acknowledge the support of Project No. VS 96049 of the Czech Ministry of Education.

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